Sheafyness of Banach Ring Spectra

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Plan

This talk is about an application of the ideas described so far. It appears in the pre-print “On the sheafyness property of spectra of Banach rings” by B. and Kremnitzer, arXiv 2009.13926

- Introduce the *problem of sheafyness* for Banach rings (and affine adic spaces).

- Propose a solution using derived (analytic) geometry.

- Compute an explicit example.

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Sheafyness problem

Let R be a non-Archimedean Banach ring.

In any reasonable study of the analytic properties of R one would like that for any $f \in R$ the localizations

$$R \to R\langle f \rangle = \frac{R\langle X \rangle}{(X - f)}$$

and

$$R \to R\langle f^{-1} \rangle = \frac{R\langle X \rangle}{(fX - 1)}$$

are admissible open localizations.

Also $\{\mathrm{Spec}^{\mathrm{an}}(R\langle f \rangle), \mathrm{Spec}^{\mathrm{an}}(R\langle f^{-1} \rangle)\}$ an admissible cover of $\mathrm{Spec}^{\mathrm{an}}(R)$.

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A consequence of this basic requirement is that the Tate-Čech complex

$$0 \to R \to R\langle f \rangle \oplus R\langle f^{-1} \rangle \to R\langle f, f^{-1} \rangle \to 0$$

must be exact for defining a structure sheaf.

Sheafyness problem

There are Banach rings for which this complex is not exact for some $f \in R$.

Until last year the story was ending here.

In 2020 our paper and Clausen-Scholze's paper "Lectures on analytic geometry" proposed similar solutions using derived geometry.

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Rough (and conjectural) picture of non-Archimedean derived analytic geometries

Classical theory		Derived theory
Geometry of Banach rings (Berkovich/Poineau/Kedlaya)	$\rightsquigarrow$	The topic of these lectures
$\cap$		$\cap$
Huber's adic spaces	$\rightsquigarrow$	Clausen-Scholze's work (Condensed mathematics)
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A non-Sheafy example

We will work out the following example, already considered by Huber and Buzzard-Verberkmoes.

Let k be a non-Archimedean field. Consider the finitely generated algebra

$$A = \frac{k[T, T^{-1}, Z]}{(Z^2)}.$$

We put on A the following norm

$$\| \sum_{n\in\mathbb{Z}, m=\{0,1\}} a_{n,m} T^n Z^m \| = \max\{\max_{n\in\mathbb{Z}}\{\rho^{-|n|}|a_{n,0}|\}, \max_{n\in\mathbb{Z}}\{\rho^{|n|}|a_{n,1}|\}\}$$

where $0 < \rho < 1$.

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Then, let us consider the Banach k-algebra $R = \widehat{(A, \|\text{-}\|)}$, obtained as the completion of A by the weird norm considered before.

As a k-vector space let us write $R = R_0 \oplus ZR_1$.

One can show that

$$R\langle T \rangle = R_0\langle T \rangle, \quad R\langle T^{-1} \rangle = R_0\langle T^{-1} \rangle.$$

Hence, the Tate-Cech complex is

$$0 \to R \to R_0\langle T \rangle \oplus R_0\langle T^{-1} \rangle \to R_0\langle T, T^{-1} \rangle \to 0$$

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A non-Sheafy example: a closer look

Notice that R is only apparently "simple". Indeed, it has the following presentation as Banach algebra

$$R \cong \varinjlim_{n \in \mathbb{N}}^{\leq 1} \frac{k\langle \rho^{-1} T, \rho T^{-1}, Z, \rho^{-1} Y_1, \rho^{-1} Y_1', \ldots, \rho^{-n} Y_n, \rho^{-n} Y_n' \rangle}{(Z^2, ZT - Y_1, ZT^{-1} - Y_1', \ldots, ZT^n - Y_n, ZT^{-n} - Y_n')}.$$

So, R is infinite dimensional as an object of analytic geometry!

Moreover, the ideal ZR is not closed and $ZR_1 = \overline{ZR}$ is not finitely generated as an R-module.

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The homotopy Zariski topology

Recall that on the homotopy category of $\mathbf{SComm}(\mathbf{Born}_k)^{\mathrm{op}}$ we defined the homotopy Zariski topology by the condition that $A \to B$ is open if

$$B \widehat{\otimes}^{\mathbb{L}}_A B \cong B$$

and the covers are given by finite families of such morphism $f_i : A \to B_i$ such the family of functor $\mathbb{L}f_i^* : \mathbf{D}^-(A) \to \mathbf{D}^-(B_i)$ is conservative.

The cover condition can be restated asking the strict exactness of the derived Tate-Čech complex

$$\mathrm{Tot}\left(0 \to A \to \prod B_i \to \prod B_i \widehat{\otimes}^{\mathbb{L}}_A B_j \to \cdots\right)$$

Issue: this topology is HUGE! No hope to describe all open subsets of any algebra.

Goal: extract a sub-topology whose localizations are "of finite type".

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Homotopical localizations

So, we would like to replace the common basic localizations by their homotopical versions.

Essentially, we would like to do the replacement

$$R\langle f\rangle = \frac{R\langle X\rangle}{(X-f)} \rightsquigarrow R\langle f\rangle^h = \mathbb{L}\frac{R\langle X\rangle}{(X-f)}$$

and

$$R\langle f^{-1}\rangle = \frac{R\langle X\rangle}{(fX-1)} \rightsquigarrow R\langle f^{-1}\rangle^h = \mathbb{L}\frac{R\langle X\rangle}{(fX-1)}.$$

And similar for the general case

$$R\left\langle \frac{f_1}{f_0},\ldots,\frac{f_n}{f_0}\right\rangle^h = \mathbb{L}\frac{R\langle X_1,\ldots X_n\rangle}{(f_0X_1-f_1,\ldots,f_0X_n-f_n)}.$$

(But we will only compute Weierstrass and Laurent localizations)

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Koszul complexes

Koszul complexes compute these homotopical quotients.

Explicitly, we will write

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$$R\langle f_1, \ldots, f_n, g_1^{-1}, \ldots, g_m^{-1}\rangle^h =$$

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Koszul complexes: sheafy case

Notice that if R is sheafy, then

$$R\langle f_1, \ldots, f_n, g_1^{-1}, \ldots, g_m^{-1}\rangle^h \cong R\langle f_1, \ldots, f_n, g_1^{-1}, \ldots, g_m^{-1}\rangle$$

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in $D^-(R)$.

So, we are generalizing the usual notion of localization.

Theorem 4.27

Suppose that R is defined over a strongly Noetherian Tate ring A. For any $f_0, f_1, \ldots, f_n \in R$, such that $(f_0, \ldots, f_n) = R$, the morphism

$$R \to R\left\langle \frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\right\rangle^h$$

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$$R \to R\left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$$

is a homotopy epimorphism.

Sketch of the proof

We summarize the main ideas of the proof:

- in general we can write $R \cong \varinjlim^{\leq 1} R_i$, where R_i are affinoid over A;
- then we can define $R_{\text{born}} \cong \varinjlim R_i$ and use the fact that $\varinjlim$ is strongly exact;
- so, it is easy to check that $R_{\text{born}} \to R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h$ is a homotopy epimorphism;
- then we do the computation

$$R_{\text{born}} \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h \widehat{\otimes}^{\mathbb{L}}_{R_{\text{born}}} R \cong R \left\langle \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right\rangle^h.$$

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Derived Tate's acyclicity

Derived Tate's acyclicity holds for standard rational covers. More precisely:

Theorem 4.29

For $f_0, f_1, \ldots, f_n \in R$, with $(f_0, \ldots, f_n) = R$, the family of morphisms

$$R \to R\left\langle \frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i} \right\rangle^h$$

is a cover for the homotopy Zariski topology.

Theorem 4.29

For any $f_0, f_1, \ldots, f_n \in R$, such that $(f_0, \ldots, f_n) = R$, the derived Tate-Cech complex

$$\mathrm{Tot}\left(0 \to R \to \prod_i R\left\langle \frac{f_0}{f_i}, .., \frac{f_n}{f_i} \right\rangle^h \to \prod_{i,j} R\left\langle \frac{f_0}{f_i}, .., \frac{f_n}{f_i} \right\rangle^h \widehat{\otimes}^{\mathbb{L}}_R R\left\langle \frac{f_0}{f_j}, .., \frac{f_n}{f_j} \right\rangle^h \to \right)$$

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The non-Sheafy example

If R is sheafy, the derived Tate's acyclicity gives back Tate's acyclicity.

Let us study what happens when $R = (\widehat{\frac{k[T, T^{-1}, Z]}{(Z^2)}}, \|.\|)$ is the non-sheafy Banach ring described before.

Proposition 5.7

The morphisms

$$R\langle X \rangle \stackrel{\mu_{(X-T)}}{\longrightarrow} R\langle X \rangle, \quad R\langle X \rangle \stackrel{\mu_{(T^{-1}X-1)}}{\longrightarrow} R\langle X \rangle$$

are monomorphisms but they are not strict.

Proof:

The series

$$\sum_{n=0}^{\infty} (-1)^{n+1} Z T^{-n+1} X^n, \quad \sum_{n=0}^{\infty} (-1)^{n+1} Z T^n X^n$$

as in $R\langle X \rangle$. This shows that $R \cap (X - T) = R \cap (T^{-1}X - 1) = ZR$, and ZR is not closed.

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This implies that

$$R\langle T\rangle^h \neq R\langle T\rangle, \quad R\langle T^{-1}\rangle^h \neq R\langle T^{-1}\rangle.$$

One can check explicitly that $R \to R\langle T\rangle^h$ and $R \to R\langle T^{-1}\rangle^h$ are homotopy epimorphisms (Proposition 5.8), but clearly $R \to R\langle T\rangle$ and $R \to R\langle T^{-1}\rangle$ are not.

Notice that

$$R\langle T\rangle^h = \frac{R\langle X\rangle}{(X-T)}, \quad R\langle T^{-1}\rangle^h = \frac{R\langle X\rangle}{(T^{-1}X-1)}$$

in $\mathrm{LH}(\mathbf{Ban}_R)$, but

$$\alpha : R \to \frac{R\langle X\rangle}{(X-T)} \oplus \frac{R\langle X\rangle}{(T^{-1}X-1)}$$

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To understand what happens we have to study $R\langle T\rangle^h \widehat{\otimes}^{\mathbb{L}}_R R\langle T^{-1}\rangle^h$.

Proposition 5.9

The following holds

$$\mathrm{LH}^n(R\langle T\rangle^h \widehat{\otimes}^{\mathbb{L}}_R R\langle T^{-1}\rangle^h) \cong \begin{cases} 0, & n \geq 2 \\ \mathrm{Ker}\,(\alpha) & n = 1 \\ \frac{R\langle X,Y\rangle}{(X-T,YT-1)} & n = 0. \end{cases}$$

So, we can compute the total complex of

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And one can check that this is exact in $\mathrm{LH}\,(\mathbf{Ban}_R)$.
In terms of objects of $\mathbf{Ban}_R$ we have that the total complex of the derived Tate-Cech complex is

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Final remarks

Our results permit to associate to R a spectrum $\mathrm{Spa}^h(R)$ where open localizations are homotopical rational localizations and covers are covers for the homotopy Zariski topology.

Theorem

The site $\mathrm{Spa}^h(R)$ is spectral and it has a derived structure sheaf of Banach algebras.

If R is defined over k there is a map $\mathrm{Spa}(R) \to \mathrm{Spa}^h(R)$.

So, one has the canonical maps $\mathrm{Spa}^r(R) \to \mathrm{Spa}(R) \to \mathrm{Spa}^h(R)$.

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A lot of generalization are possible (some already done):

- R can be taken to be a (nice) bonological algebra (this is done in the paper);
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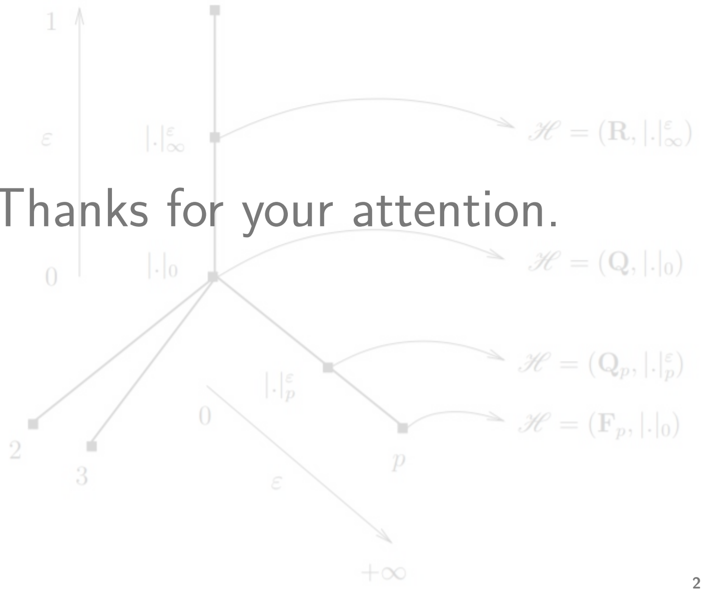
Thanks



Thanks for your attention.