HAG contexts for analytic geometry

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Plan

- Briefly recall what a HAG context is.

- Define a HAG context using bornological spaces/modules.

- Explain how this HAG context relates with algebraic geometry, analytic geometry, differential geometry, topology, and more...

- Maybe some applications, if time permits.

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HAG contexts

Reference

Toën, Bertrand, and Vezzosi, Gabriele. "Homotopical algebraic geometry I", "Homotopical algebraic geometry II".

Idea

HAG contexts are categories with a bunch of properties that make them suitable for derived/homotopical geometry.

Let us explain this with examples.

Algebraic geometry can be presented as follows.
Starting with the category $\mathbf{Ab} = \mathbf{Mod}_{\mathbb{Z}}$ of abelian groups, consider the tensor product $\otimes_{\mathbb{Z}}$ and the category commutative monoids for the tensor product $\mathbf{Comm}(\mathbf{Mod}_{\mathbb{Z}})$, a.k.a. commutative rings.

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On affine schemes $\mathbf{Comm}(\mathbf{Mod}_{\mathbb{Z}})^{\mathrm{op}}$ one considers various topologies.

Notice that most of these topologies can be defined using only categorical concepts of $\mathbf{Comm}(\mathbf{Mod}_{\mathbb{Z}})$.

For example, a *localization* in $\mathbf{Comm}(\mathbf{Mod}_{\mathbb{Z}})$ is a flat epimorphism of finite presentation. This gives the Zariski open immersions on $\mathbf{Comm}(\mathbf{Mod}_{\mathbb{Z}})^{\mathrm{op}}$.

Similarly, on $\mathbf{Comm}(\mathbf{Mod}_{\mathbb{Z}})^{\mathrm{op}}$ one can consider the étale, fppf, fpqc, h, Nisnevich topologies etc...

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Observation

Given any category $\mathbf{C}$ and a monoidal structure on $\otimes$ I can mimic this situation and define schemes, stack, and all basic constructions of Algebraic Geometry on $\mathbf{Comm}(\mathbf{C})^{\mathrm{op}}$.

For example, $\mathbf{Comm}((\mathbf{Sets}, \prod)) =$ commutative monoids. (Roughly, toric varieties)

Another example is $(D(\mathbf{Mod}_R), \otimes_R^{\mathbb{L}})$, for R a commutative ring.

But working directly with $D(\mathbf{Mod}_R)$ is not easy. So, it is common practice to present it as the homotopy category of a model category of an ∞-category.

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The presentation of a triangulated category as the homotopy category of an ∞-category can be done in many ways.

In this talk we use model categories to present the homotopy categories.

(Simplified definition of model category.)

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Hovey, Mark. "Model categories". No. 63. American Mathematical Soc., 2007.

Model category

A *model category* $\mathbf{C}$ is a category $\mathbf{C}$ with extra data of classes of morphisms $W, F, C \subset \mathbf{C}$ such that

- $\mathbf{C}$ has all limits and colimits;
- the classes of morphisms W, F, C satisfy a bunch of axioms. The most important class is W, the class of *weak equivalences*.

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Why W is important? Because the homotopy category $\mathbf{Ho}(\mathbf{C}) = \mathbf{C}[W^{-1}]$.

The main example is $\mathbf{Ch}(\mathbf{Mod}_R)$ where the class of weak equivalences is the class of quasi-isomorphisms. Indeed, $D(\mathbf{Mod}_R) = \mathbf{Ch}(\mathbf{Mod}_R)[\mathrm{qis}^{-1}]$.

The rest of the structure $F, C \subset \mathbf{C}$, called cofibrations and fibrations, are analogous to the projective and injective resolutions in $D(\mathbf{Mod}_R)$. In this context one speaks of fibrant and cofibrant replacement and of fibrant and cofibrant objects.

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HAG context

A *HAG context* is model category $\mathbf{C}$ equipped with a monoidal structure $\otimes$ that satisfies the (commutative) monoid axioms.

This (roughly) asks that $\otimes$ is derivable to a bifunctor $\otimes^{\mathbb{L}} : \mathbf{Ho}(\mathbf{C}) \times \mathbf{Ho}(\mathbf{C}) \to \mathbf{Ho}(\mathbf{C})$, that cofibrant replacement $Q\mathrm{Id} \to \mathrm{Id}$ of the identity are sent to weak-equivalences $Q\mathrm{Id} \otimes X \to \mathrm{Id} \otimes X$, for all X, plus some more conditions.

This implies that $\mathbf{Comm}(\mathbf{C})$ has a nice model structure induced by the one of $\mathbf{C}$, that $\mathbf{Ho}(\mathbf{C})$ is symmetric monoidal and that $\mathbf{Ho}(\mathbf{Comm}(\mathbf{C})) \cong \mathbf{Comm}(\mathbf{Ho}(\mathbf{C}))$.

Examples

- $\mathbf{Ch}^{\geq 0}(\mathbf{Mod}_R)$ satisfies the monoid axioms if $\mathrm{char}\,(R) = 0$.
- $\mathbf{Ch}^{\geq 0}(\mathbf{Mod}_R)$ does not satisfy the monoid axioms if $\mathrm{char}\,(R) \neq 0$.
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Basic use of an HAG contexts

It is possible to use all of this for defining geometric spaces over **Ho**(**Comm**(**C**)).

I describe the simplest possible topology on **Ho**(**Comm**(**C**)$^{\mathrm{op}}$).

Homotopy Zariski localization

A $f : A \to B$ in **Ho**(**Comm**(**C**)) is called a *(formal) homotopy Zariski localization* if it is a epimorphism.

Usually people write the condition

$$B \otimes_A^{\mathbb{L}} B \xrightarrow{\cong} B$$

and call it a *homotopy epimorphism*.

If A and B are rings and f is a finite type, then this is equivalent to Zariski localization.

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Homotopy Zariski cover

A finite family $\{f_i : A \to B_i\}$ determine a cover in $\mathbf{Ho}(\mathbf{Comm}(\mathbf{C})^{\mathrm{op}})$ if the pullback functors $\mathbb{L}f_i^* : \mathbf{Ho}(\mathbf{Mod}_A) \to \mathbf{Ho}(\mathbf{Mod}_{B_i})$ are a conservative family of functors.

Again coincides with the usual notion.

Then, stacks are defined as pre-sheaves $\mathscr{F} : \mathbf{Ho}(\mathbf{Comm}(\mathbf{C})) \to \mathbf{Simp}(\mathbf{Sets})$ that satisfy descent, i.e.

$$\mathscr{F}(X) \cong \underleftarrow{\mathrm{holim}}\left(\prod \mathscr{F}(X_i) \to \prod \mathscr{F}(X_i \times_X X_j) \to \cdots\right)$$

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In algebraic geometry it is clear what is the basic HAG context to work with: $(\mathbf{Ch}^{\leq 0}(\mathbf{Mod}_R), \otimes_R)$ where equivalences are quasi-isomorphisms (or better $\mathbf{Simp}(\mathbf{Mod}_R)$).

But in other situations?

For example, analytic spaces are build from models that are spectra of Banach or Fréchet algebras. What's a HAG context here?

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In algebraic geometry it is clear what is the basic HAG context to work with: $(\mathbf{Ch}^{\leq 0}(\mathbf{Mod}_R), \otimes_R)$ where equivalences are quasi-isomorphisms (or better $\mathbf{Simp}(\mathbf{Mod}_R)$).

But in other situations?

For example, analytic spaces are build from models that are spectra of Banach or Fréchet algebras. What's a HAG context here?

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The main issue is that one would like to work with complexes of topological modules or similar structures, but these categories are not abelian.

For dealing with this kind of situations Quillen invented the theory of exact categories.

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Exact categories

Let **C** be an additive category (small if you want).

Suppose that **C** comes with a fully faithful embedding $\iota : \mathbf{C} \to \mathbf{A}$ into an abelian category **A** and that **C** is stable by extensions in **A**.

Let us denote by S the class of short exact sequences in **A**.

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An *exact category* **C** is an additive category together a class of kernel-cokernel pairs $\mathcal{E}$ such that $\mathcal{E} = \iota^{-1}(S)$.

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In general $\mathcal{E}$ amount to extra data, it not intrinsic in $\mathbf{C}$. But with mild assumptions on $\mathbf{C}$ (weakly idempotent complete), $\mathbf{C}$ has a biggest exact structure $\mathcal{E}_{\max}$.

In any case we can associate the derived category $D((\mathbf{C},\mathcal{E}))$.

Quasi-abelian category

In the case when $\mathbf{C}$ is pre-abelian and $\mathcal{E}_{\max} = \mathcal{E}_{strict}$ then $\mathbf{C}$ is called *quasi-abelian*.

Recall that in any pre-abelian category one has the factorization

$$\begin{array}{ccccccccc} 0 \longrightarrow \operatorname{Ker}(f) \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow \operatorname{Coker}(f) \longrightarrow 0 \\ & \downarrow & & \uparrow & \\ & \operatorname{Coim}(f) & \longrightarrow & \operatorname{Im}(f) & \end{array}$$

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Quasi-abelian category	$\mathcal{E}_{strict}$	$D(\mathbf{C})$ + 2 t-structures close to each other
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$$D(\mathbf{C}) = \frac{K(\mathbf{C})}{N}$$

$N \subset K(\mathbf{C})$ is the thick subcategory of strictly exact complexes.

This is equivalent to invert chain morphisms whose cone is strictly exact.

On $D(\mathbf{C})$ there are the left and right t-structures whose hearts are $\mathrm{LH}(\mathbf{C})$ and $\mathrm{RH}(\mathbf{C})$.

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It is useful to know that:

- $D(\mathbf{C}) \cong D(\mathrm{LH}(\mathbf{C})) \cong D(\mathrm{RH}(\mathbf{C}))$;
- $\mathbf{C} \to \mathrm{LH}(\mathbf{C})$ reflective;
- $\mathbf{C} \to \mathrm{RH}(\mathbf{C})$ coreflective.

Exact functor

Let $F : \mathbf{C} \to \mathbf{D}$ be an additive functor between quasi-abelian categories, then F is said

- *left exact* if it preserves kernels of strict morphisms;
- *strongly left exact* if it preserves kernels of all morphisms;
- You can guess (resp. *strongly*) *right exact*;
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And these notions are related with exactness with respect to the left and right t-structures.

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To study how far $F : \mathbf{C} \to \mathbf{D}$ is from a (strongly) exact functor we can use its derived functors $\mathbb{L}F : D^-(\mathbf{C}) \to D^-(\mathbf{D})$ and $\mathbb{R}F : D^+(\mathbf{C}) \to D^+(\mathbf{D})$, when they exists.

It can be proved that $\mathbb{L}F$ and/or $\mathbb{R}F$ exist when $\mathbf{C}$ has enough acyclic objects for F.

Examples

- Abelian categories are quasi-abelian.
- Topological abelian groups.
- Locally convex spaces over $\mathbb{R}$.
- If $\mathbf{C}$ is nice enough (*elementary*), $\mathbf{Sh}_{\mathbf{C}}(X)$ is quasi-abelian.

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The Dold-Kan correspondence is true also for exact categories.

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Banach abelian groups

Banach abelian group

A *Banach abelian group* is an abelian group M together with a norm function $|\cdot|_M : M \to \mathbb{R}_{\geq 0}$ such that

- $|x|_M = 0$ iff $x = 0$;
- $|x+y|_M \leq |x|_M + |y|_M$;
- M is complete with respect to the distance induced by $|\cdot|_M$.

A group homomorphism $f : (M, |\cdot|_M) \to (N, |\cdot|_N)$ is *bounded* if $\exists C \geq 0$ such that

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Proposition

Let $f : M \to N$ be a morphism in **BanAb**, then

- $\mathrm{Ker}\,(f) = f^{-1}(0)$ with the restriction of the norm of M.
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- f is a monomorphism if and only if it is injective.
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From this it is possible to verify that the category **BanAb** is quasi-abelian.

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- $\operatorname{Coim}(f) = \frac{M}{f^{-1}(0)}$ with the quotient norm.
- f is a monomorphism if and only if it is injective.
- f is a epimorphism if and only if it has dense image.

From this it is possible to verify that the category **BanAb** is quasi-abelian.

Banach abelian groups

Proposition

Let $f : M \to N$ be a morphism in **BanAb**, then

- $\mathrm{Ker}\,(f) = f^{-1}(0)$ with the restriction of the norm of M.
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Banach rings

Projective tensor product

Given $M, N \in \mathbf{BanAb}$ on $M \otimes_{\mathbb{Z}} N$ define the norm

$$|x| = \inf \left\{ \sum |a_i|_M |b_i|_N \middle| x = \sum a_i \otimes b_i \right\}$$

and the completion

$$M \widehat{\otimes}_{\mathbb{Z}} N = \widehat{M \otimes_{\mathbb{Z}} N}.$$

The monoidal structure $\widehat{\otimes}_{\mathbb{Z}}$ has a right adjoint, given by $\mathrm{Hom}(\cdot, \cdot)$ equipped with the operator norm.

It is easy to check that $\mathbf{Comm}((\mathbf{BanAb}, \widehat{\otimes}_{\mathbb{Z}}))$ is the usual category of commutative Banach rings, *i.e.* commutative rings equipped with a submultiplicative norm and bounded homomorphisms.

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Banach modules

Examples of Banach rings

- $(\mathbb{Z}, |\cdot|_\infty)$ with the Euclidean norm.
- $(\mathbb{Z}, |\cdot|_0)$ with the trivial norm.
- $(\mathbb{C}, |\cdot|_\infty)$ with the Euclidean norm.
- $(k, |\cdot|)$ non-Archimedean field.

For any Banach ring R denote with $\mathbf{Ban}_R$ the category of Banach R-modules.

Proposition

- For any Banach ring R, $\mathbf{Ban}_R$ is quasi-abelian.
- $\mathbf{BanAb} \cong \mathbf{Ban}_{(\mathbb{Z}, |\cdot|_\infty)}$.
- $\mathbf{Ban}_R$ is closed symmetric monoidal too, with $\widehat{\otimes}_R$

$$M \widehat{\otimes}_R N = \mathrm{Coker}\,(M \widehat{\otimes}_{\mathbb{Z}} R \widehat{\otimes}_{\mathbb{Z}} N \to M \widehat{\otimes}_{\mathbb{Z}} N).$$

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Projective/injective object

An object $X \in \mathbf{Ban}_R$ is said *projective* (resp. *injective*) if the functor $\mathrm{Hom}(X, \cdot)$ is exact (resp. $\mathrm{Hom}(\cdot, X)$).

Enough projective/injective objects

We say that $\mathbf{Ban}_R$ has enough *projective* (resp. *injective*) *objects* if for any object $X \in \mathbf{Ban}_R$ there exists a projective P and a strict epimorphism $P \to X$ (resp. an injective I and a strict monomorphism $X \to I$).

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$\mathbf{Ban}_R$ has enough projective objects.

Proof.

ℓ^1-modules are projective and every object as a strict epimorphism from an ℓ^1-module.

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Proposition

$\mathbf{Ban}_R$ has enough injective objects sometimes (for example for R a complete valued field).

We have seen that $\mathbf{Ban}_R$ has very nice properties.

But does not have any infinite product or infinite coproduct!

And many objects of interest, like Fréchet spaces, are not in this category.

The most common solution to this problem, over a base field $(k, |\cdot|)$, is to consider the category of locally convex spaces $\mathbf{Loc}_k$. But:

- $\mathbf{Loc}_k$ does not admit a closed monoidal structure.
- Objects of $\mathbf{Loc}_k$ are not complete.
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Bornological modules

Classically a bornology on a vector space over $(k, |\cdot|)$ is defined using the notion of bounded subset.

Problem

The classical definition using bounded subset does not work over any Banach ring.

For example $(\mathbb{Z}, |\cdot|_\infty) \to (\mathbb{Z}, |\cdot|_\infty^\epsilon)$, $0 < \epsilon < 1$ identify the underlying "naive bornological groups".

Solution

A more abstract approach works.

If k is a valued field, than $\mathbf{Born}_k \subset \mathbf{Ind}(\mathbf{Ban}_k)$, and the essential image is the class of essentially monomorphic ind-objects.

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The category of *(complete) bornological modules* over R is the subcategory $\mathbf{Born}_R \subset \mathbf{Ind}(\mathbf{Ban}_R)$ of essentially monomorphic objects.

This always makes sense and contains $\mathbf{Ban}_R$ fully faithfully.

Proposition

- $\mathbf{Born}_R$ is quasi-abelian.
- $\mathbf{Born}_R$ has all limits and colimits.
- $\mathbf{Born}_R$ closed symmetric monodial
  $\text{“}\varinjlim\text{”}_{i\in I} M_i \widehat{\otimes}_R \text{“}\varinjlim\text{”}_{j\in J} N_j \cong \text{“}\varinjlim\text{”}_{I\times J} M_i \widehat{\otimes}_R N_j$.
- $\mathbf{Born}_R$ has strongly exact filtered direct limits.
- $\mathbf{Born}_R$ has enough projectives $\bigoplus_{i\in I} P_i$, P_i projectives in $\mathbf{Ban}_R$.
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- $\mathbf{Born}_R$ is quasi-abelian.
- $\mathbf{Born}_R$ has all limits and colimits.
- $\mathbf{Born}_R$ closed symmetric monodial
  $\text{``}\varinjlim\text{''}_{i \in I} M_i \widehat{\otimes}_R \text{``}\varinjlim\text{''}_{j \in J} N_j \cong \text{``}\varinjlim\text{''}_{I \times J} M_i \widehat{\otimes}_R N_j$.
- $\mathbf{Born}_R$ has strongly exact filtered direct limits.
- $\mathbf{Born}_R$ has enough projectives $\bigoplus_{i \in I} P_i$, P_i projectives in $\mathbf{Ban}_R$.
- $\mathbf{Born}_R$ does not have enough injectives.

Bornological modules: examples

We give some examples of bornological modules.

Examples

- If $R = k$ a valued field, $\mathbf{Born}_k$ is the classical category of (complete) bornological spaces.
- Consider $(\mathbb{Z}, |\cdot|_\infty^\epsilon)$ with $0 < \epsilon \leq 1$, then $\varinjlim_{\epsilon \in (0,1]} (\mathbb{Z}, |\cdot|_\infty^\epsilon)$ is a bornological module.

Over a valued field k there is a functor

$$(\cdot)^b : \mathbf{CLoc}_k \to \mathbf{Born}_k$$

sending a complete locally convex spaces to the bornological space determined by its bounded subsets.

$(\cdot)^b$ restricts to an equivalence for Fréchet spaces.

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Bornological rings

Bornological ring

The category of *bornological rings* is $\textbf{Comm}((\textbf{Born}_{(\mathbb{Z},|\cdot|_\infty)}, \widehat{\otimes}_\mathbb{Z}))$.

Examples

- Let k be a valued field and R a bornological algebra over k, then R is a bornological ring.
- Let k be a valued field and R a Fréchet algebra over k, then R is a bornological ring.
- Algebras of analytic functions are bornological algebras.
- Adic rings are bornological algebras.
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For any bornological ring R the category $\mathbf{Ch}^{\leq 0}(\mathbf{Born}_R) = \mathbf{Simp}(\mathbf{Born}_R)$ is a HAG context.

In particular we can consider on $\mathbf{Ho}(\mathbf{Simp}(\mathbf{Born}_R))$ the homotopy Zariski topology and see what we get.

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Relations with Algebraic Geometry

Let R be a bornological ring and $M \in \mathbf{Mod}_R$. We can put on M the strongest bornology compatible with the one of R, denoted $M_{\text{fine}} = \varinjlim_{M' \subset M, f.p.} M'$.

Proposition

The functor $(\text{-})_{\text{fine}} : \mathbf{Mod}_R \to \mathbf{Born}_R$ is fully faithful and strongly monoidal. In particular for any morphism of R-algebras $A \to B$ we have that

$$B \otimes_A^{\mathbf{L}} B \cong B \iff B_{\text{fine}} \widehat{\otimes}_{A_{\text{fine}}}^{\mathbf{L}} B_{\text{fine}} \cong B_{\text{fine}}.$$

So, the homotopy Zariski topology of $\mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_R))^{\text{op}}$ restricts to the usual homotopy Zariski topology on $\mathbf{Comm}(\mathbf{Simp}(\mathbf{Mod}_R))^{\text{op}}$.

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Consider k a non-Archimedean valued field.

On k there is the category of affinoid algebras, $\mathbf{Aff}_k$. These are Banach algebras of convergent power-series in finitely many variables over k.

In particular, $\mathbf{Aff}_k \subset \mathbf{Comm}(\mathbf{Ban}_k) \subset \mathbf{Comm}(\mathbf{Born}_k)$.

Affinoid spaces are just $\mathbf{Aff}_k^{\mathrm{op}}$. On $\mathbf{Aff}_k^{\mathrm{op}}$ Tate defined the so-called G-topology of rigid spaces.

Analytic spaces over k are defined by glueing affinoid spaces via the G-topology.

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Theorem (For rigid spaces)

The homotopy Zariski topology on $\mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_k))^{\mathrm{op}}$ restricts to the G-topology, via the inclusion $\mathbf{Aff}_k^{\mathrm{op}} \subset \mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_k))^{\mathrm{op}}$.

There are other versions of this theorem.

Theorem (Stein version)

The homotopy Zariski topology on $\mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_k))^{\mathrm{op}}$ restricts to the standard topology (but finite covers!), via the inclusion $\mathbf{Stein}_k^{\mathrm{op}} \subset \mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_k))^{\mathrm{op}}$.

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Relations with Differential Geometry

The algebras $C^\infty(M)$ are Fréchet algebras. In particular they belong to **Comm**(**Born**$_{\mathbb{R}}$) and the association $M \mapsto C^\infty(M)$ is an anti-equivalence.

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The homotopy Zariski topology on **Comm**(**Simp**(**Born**$_{\mathbb{R}}$))$^{\mathrm{op}}$ restricts to the standard topology (but finite covers!), via the inclusion **Smooth** $\subset$ **Comm**(**Simp**(**Born**$_{\mathbb{R}}$))$^{\mathrm{op}}$.

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Of course also commutative C^*-algebras lie in $\mathbf{Comm}(\mathbf{Born}_{\mathbb{C}})$.

Theorem (C^*-algebra version)

The homotopy Zariski topology on $\mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_{\mathbb{C}}))^{\mathrm{op}}$ restricts to the topology given by finite coverings of topological embeddings, via the inclusion $\mathbf{CH} \subset \mathbf{Comm}(\mathbf{Simp}(\mathbf{Born}_{\mathbb{C}}))^{\mathrm{op}}$.

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If R is a Banach ring, then there is not clear notion of $\mathrm{Spec}\,(R)$.

The problem is that with all proposed notions the pre-structure sheaf fails to be a sheaf.

(It is actually impossible to put any reasonable notion of structure sheaf on any reasonable notion of spectrum)

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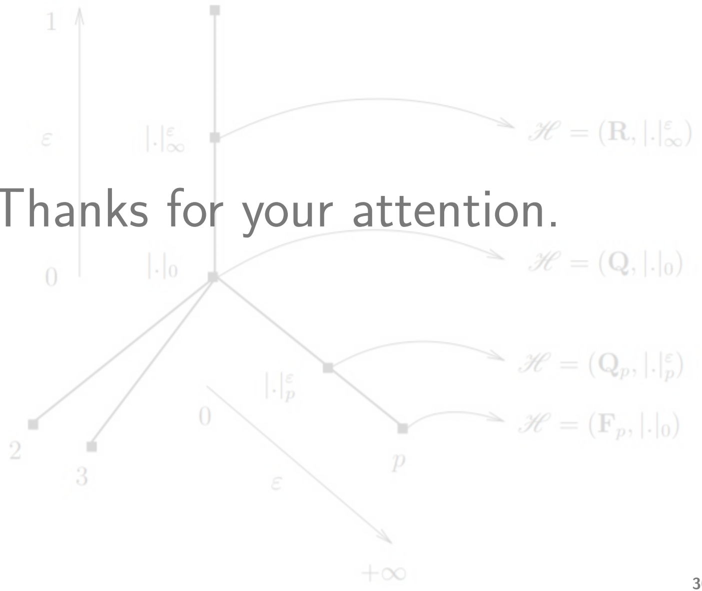

Thanks for your attention.