Homological Bornological Algebra

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Plan

- Brief introduction to the theory of quasi-abelian categories.

- Definition of bornological algebras and bornological modules over them.

- Some examples.

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Exact categories

A nice reference for the theory of exact categories is Bühler, *Exact Categories*, Expositiones Mathematicae, Volume 28, 2010.

Let $\mathbf{C}$ be an additive category (small if you want).

Suppose that $\mathbf{C}$ comes with a fully faithful embedding $\iota : \mathbf{C} \to \mathbf{A}$ into an abelian category $\mathbf{A}$ and that $\mathbf{C}$ is stable by extensions in $\mathbf{A}$.

Let us denote by S the class of short exact sequences in $\mathbf{A}$.

Exact category

An *exact category* $\mathbf{C}$ is an additive category together a class of kernel-cokernel pairs $\mathcal{E}$ such that $\mathcal{E} = \iota^{-1}(S)$.

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Exact categories

In general $\mathcal{E}$ amount to extra data, it not intrinsic in **C**. But with mild assumptions on **C** (weakly idempotent complete), **C** has a biggest exact structure $\mathcal{E}_{\mathsf{max}}$.

In any case we can associate the derived category $D((\mathbf{C}, \mathcal{E}))$.

Quasi-abelian category

In the case when **C** is pre-abelian and $\mathcal{E}_{\mathsf{max}} = \mathcal{E}_{strict}$ then **C** is called *quasi-abelian*.

Recall that in any pre-abelian category one has the factorization

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Ker}\,(f) & \longrightarrow & X & \xrightarrow{\quad f \quad} & Y & \longrightarrow & \mathrm{Coker}\,(f) & \longrightarrow & 0 \\ & & & & \downarrow & & \uparrow & & & & \\ & & & & \mathrm{Coim}\,(f) & \longrightarrow & \mathrm{Im}\,(f) & & & & \end{array}$$

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In general $\mathcal{E}$ amount to extra data, it not intrinsic in $\mathbf{C}$. But with mild assumptions on $\mathbf{C}$ (weakly idempotent complete), $\mathbf{C}$ has a biggest exact structure $\mathcal{E}_{\max}$.

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One has the following hierarchy of "homological complexity":

C	$\mathcal{E}$	"="
Abelian category	$\mathcal{E}_{strict}$	$D(\mathbf{C})$ + t-structure
Quasi-abelian category	$\mathcal{E}_{strict}$	$D(\mathbf{C})$ + 2 t-structures close to each other
Pre-abelian category	$\mathcal{E}$	$D(\mathbf{C})$ + 2 t-structures not so close
Other "nice" exact category	$\mathcal{E}$	$D(\mathbf{C})$ + 2 t-structures even less closer
General exact category	$\mathcal{E}$	$D(\mathbf{C})$ + 2 t-structures without any relation

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Schneiders J.-P. "*Quasi-abelian categories and sheaves*", Société mathématique de France, 1999.

For $\mathbf{C}$ quasi-abelian the category

$$D(\mathbf{C}) = \frac{K(\mathbf{C})}{N}$$

$N \subset K(\mathbf{C})$ is the thick subcategory of strictly exact complexes.

This is equivalent to invert chain morphisms whose cone is strictly exact.

On $D(\mathbf{C})$ there are the left and right t-structures whose hearts are $\mathrm{LH}(\mathbf{C})$ and $\mathrm{RH}(\mathbf{C})$.

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Quasi-abelian categories

It is useful to know that:

- $D(\mathbf{C}) \cong D(\mathrm{LH}(\mathbf{C})) \cong D(\mathrm{RH}(\mathbf{C}))$;
- $\mathbf{C} \to \mathrm{LH}(\mathbf{C})$ reflective;
- $\mathbf{C} \to \mathrm{RH}(\mathbf{C})$ coreflective.

Exact functor

Let $F : \mathbf{C} \to \mathbf{D}$ be an additive functor between quasi-abelian categories, then F is said

- *left exact* if it preserves kernels of strict morphisms;
- *strongly left exact* if it preserves kernels of all morphisms;
- You can guess (resp. *strongly*) *right exact*;
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And these notions are related with exactness with respect to the left and right t-structures.

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To study how far $F : \mathbf{C} \to \mathbf{D}$ is from a (strongly) exact functor we can use its derived functors $\mathbb{L}F : D^-(\mathbf{C}) \to D^-(\mathbf{D})$ and $\mathbb{R}F : D^+(\mathbf{C}) \to D^+(\mathbf{D})$, when they exists.

It can be proved that $\mathbb{L}F$ and/or $\mathbb{R}F$ exist when $\mathbf{C}$ has enough acyclic objects for F.

Examples

- Abelian categories are quasi-abelian.
- Topological abelian groups.
- Locally convex spaces over $\mathbb{R}$.
- Filtered abelian groups.
- If $\mathbf{C}$ is nice enough (*elementary*), $\mathbf{Sh}_{\mathbf{C}}(X)$ is quasi-abelian.

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Banach abelian groups

Banach abelian group

A *Banach abelian group* is an abelian group M together with a norm function $|\cdot|_M : M \to \mathbb{R}_{\geq 0}$ such that

- $|x|_M = 0$ iff $x = 0$;
- $|x + y|_M \leq |x|_M + |y|_M$;
- M is complete with respect to the distance induced by $|\cdot|_M$.

A group homomorphism $f : (M, |\cdot|_M) \to (N, |\cdot|_N)$ is *bounded* if $\exists C \geq 0$ such that

$$|f(x)|_N \leq C|x|_M, \quad \forall x \in M.$$

BanAb is the category of Banach abelian groups.

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Proposition

Let $f : M \to N$ be a morphism in **AbBan**, then

- $\mathrm{Ker}\,(f) = f^{-1}(0)$ with the restriction of the norm of M.
- $\mathrm{Coker}\,(f) = \frac{N}{f(M)}$ with the quotient norm.
- $\mathrm{Im}\,(f) = \overline{f(M)}$ with the norm induced by N.
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- f is a monomorphism if and only if it is injective.
- f is a epimorphism if and only if it has dense image.

From this it is possible to verify that the category **AbBan** is quasi-abelian.

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- $\mathrm{Im}\,(f) = \overline{f(M)}$ with the norm induced by N.
- $\mathrm{Coim}\,(f) = \frac{M}{f^{-1}(0)}$ with the quotient norm.
- f is a monomorphism if and only if it is injective.
- f is a epimorphism if and only if it has dense image.

From this it is possible to verify that the category **AbBan** is quasi-abelian.

Banach abelian groups

Proposition

Let $f : M \to N$ be a morphism in **AbBan**, then

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Banach rings

Projective tensor product

Given $M, N \in \mathbf{BanAb}$ on $M \otimes_{\mathbb{Z}} N$ define the norm

$$|x| = \inf\{\sum |a_i|_M |b_i|_N | x = \sum a_i \otimes b_i\}$$

and the completion

$$M \widehat{\otimes}_{\mathbb{Z}} N = \widehat{M \otimes_{\mathbb{Z}} N}.$$

The monoidal structure $\widehat{\otimes}_{\mathbb{Z}}$ has a right adjoint, given by $\mathrm{Hom}(\cdot, \cdot)$ equipped with the operator norm.

It is easy to check that $\mathbf{Comm}((\mathbf{BanAb}, \widehat{\otimes}_{\mathbb{Z}}))$ is the usual category of commutative Banach rings, *i.e.* commutative rings equipped with a submultiplicative norm and bounded homomorphisms.

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Banach modules

Examples of Banach rings

- $(\mathbb{Z}, |\cdot|_\infty)$ with the Euclidean norm.
- $(\mathbb{Z}, |\cdot|_0)$ with the trivial norm.
- $(\mathbb{C}, |\cdot|_\infty)$ with the Euclidean norm.
- $(k, |\cdot|)$ non-Archimedean field.

For any Banach ring R denote with $\mathbf{Ban}_R$ the category of Banach R-modules.

Proposition

- For any Banach ring R, $\mathbf{Ban}_R$ is quasi-abelian.
- $\mathbf{BanAb} \cong \mathbf{Ban}_{(\mathbb{Z}, |\cdot|_\infty)}$.
- $\mathbf{Ban}_R$ is closed symmetric monoidal too, with $\widehat{\otimes}_R$

$$M \widehat{\otimes}_R N = \operatorname{Coker}(M \widehat{\otimes}_\mathbb{Z} R \widehat{\otimes}_\mathbb{Z} N \to M \widehat{\otimes}_\mathbb{Z} N).$$

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Non-Archimedean and contracting Banach modules

Contracting category

The *contracting category* $\mathbf{Ban}_R^{\leq 1} \subset \mathbf{Ban}_R$ has the same objects and morphisms $f : M \to N$ such that

$$|f(m)| \leq |m|.$$

Warning: $\mathbf{Ban}_R^{\leq 1}$ is not quasi-Abelian.

Non-Archimedean category

If R is non-Archimedean then $\mathbf{Ban}_R^{\mathrm{na}} \subset \mathbf{Ban}_R$ full subcategory of ultrametric modules.

- $(\mathbf{Ban}_R^{\mathrm{na}})^{\leq 1}$ is quasi-abelian.
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Projective/injective object

An object $X \in \mathbf{Ban}_R$ is said *projective* (resp. *injective*) if the functor $\mathrm{Hom}\,(X, \cdot)$ is exact (resp. $\mathrm{Hom}\,(\cdot, X)$).

Enough projective/injective objects

We say that $\mathbf{Ban}_R$ has enough *projective* (resp. *injective*) *objects* if for any object $X \in \mathbf{Ban}_R$ there exists a projective P and a strict epimorphism $P \to X$ (resp. an injective I and a strict monomorphism $X \to I$).

Proposition

$\mathbf{Ban}_R$ has enough projective objects.

Proof.

ℓ^1-modules are projective and every object as a strict epimorphism from an ℓ^1-module.

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Proposition

$\mathbf{Ban}_R$ has enough injective objects sometimes (for example for R a spherically complete valued field).

We have seen that $\mathbf{Ban}_R$ has very nice properties.

But does not have any infinite product or infinite coproduct!

And many objects of interest, like Fréchet spaces, are not in this category.

The most common solution to this problem, over a base field $(k, |\cdot|)$, is to consider the category of locally convex spaces $\mathbf{Loc}_k$. But:

- $\mathbf{Loc}_k$ does not admit a closed monoidal structure.
- Objects of $\mathbf{Loc}_k$ are not complete.
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Bornological modules

Jack introduced what a bornology on a vector space over $(k, |\cdot|)$.

Problem

The classical definition using bounded subset does not work over any Banach ring.

For example $(\mathbb{Z}, |\cdot|_\infty) \to (\mathbb{Z}, |\cdot|_\infty^\epsilon)$, $0 < \epsilon < 1$ identify the underlying "naive bornological groups".

Solution

A more abstract approach works.

If k is a valued field, than $\mathbf{Born}_k \subset \mathbf{Ind}(\mathbf{Ban}_k)$, and the essential image is the class of essentially monomorphic ind-objects.

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The category of *(complete) bornological modules* over R is the subcategory $\mathbf{Born}_R \subset \mathbf{Ind}(\mathbf{Ban}_R)$ of essentially monomorphic objects.

This always makes sense and contains $\mathbf{Ban}_R$ fully faithfully.

Proposition

- $\mathbf{Born}_R$ is quasi-abelian.
- $\mathbf{Born}_R$ has all limits and colimits.
- $\mathbf{Born}_R$ closed symmetric monodial $\text{“}\varinjlim\text{”}_{i\in I} M_i \widehat{\otimes}_R \text{“}\varinjlim\text{”}_{j\in J} N_j \cong \text{“}\varinjlim\text{”}_{I\times J} M_i \widehat{\otimes}_R N_j$.
- $\mathbf{Born}_R$ has strongly exact filtered direct limits.
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The category of *(complete) bornological modules* over R is the subcategory $\mathbf{Born}_R \subset \mathbf{Ind}(\mathbf{Ban}_R)$ of essentially monomorphic objects.

This always makes sense and contains $\mathbf{Ban}_R$ fully faithfully.

Proposition

- $\mathbf{Born}_R$ is quasi-abelian.
- $\mathbf{Born}_R$ has all limits and colimits.
- $\mathbf{Born}_R$ closed symmetric monodial
  $\text{“}\varinjlim\text{”}_{i\in I} M_i \widehat{\otimes}_R \text{“}\varinjlim\text{”}_{j\in J} N_j \cong \text{“}\varinjlim\text{”}_{I\times J} M_i \widehat{\otimes}_R N_j$.
- $\mathbf{Born}_R$ has strongly exact filtered direct limits.
- $\mathbf{Born}_R$ has enough projectives $\bigoplus_{i\in I} P_i$, P_i projectives in $\mathbf{Ban}_R$.
- $\mathbf{Born}_R$ does not have enough injectives.

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Bornological modules: examples

We give some examples of bornological modules.

Examples

- If $R = k$ a valued field, $\mathbf{Born}_k$ is the classical category of (complete) bornological spaces.
- Consider $(\mathbb{Z}, |\cdot|_\infty^\epsilon)$ with $0 < \epsilon \leq 1$, then $\varinjlim_{\epsilon \in (0,1]} (\mathbb{Z}, |\cdot|_\infty^\epsilon)$ is a bornological module.

Over a valued field k there is a functor

$$(\cdot)^b : \mathbf{CLoc}_k \to \mathbf{Born}_k$$

sending a complete locally convex spaces to the bornological space determined by its bounded subsets.

$(\cdot)^b$ restricts to an equivalence for Fréchet spaces.

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Bornological rings

Bornological ring

The category of *bornological rings* is $\mathbf{Comm}((\mathbf{Born}_{(\mathbb{Z},|\cdot|_\infty)}, \widehat{\otimes}_{\mathbb{Z}}))$.

Examples

- Let k be a valued field and R a bornological algebra over k, then R is a bornological ring.
- Let k be a valued field and R a Fréchet algebra over k, then R is a bornological ring.
- Algebras of analytic functions are bornological algebras.
- Adic rings are bornological algebras.
- The bornological module $\varinjlim_{\epsilon \in (0,1]} (\mathbb{Z}, |\cdot|_\infty^\epsilon)$ is a bornological ring.

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Bornological modules and ind-banach

We defined $\mathbf{Born}_R$ as a full subcategory of $\mathbf{Ind}(\mathbf{Ban}_R)$.

Proposition

The inclusion $\mathbf{Born}_R \subset \mathbf{Ind}(\mathbf{Ban}_R)$ induces an equivalence of tensor triangulated categories $D(\mathbf{Born}_R) \cong D(\mathbf{Ind}(\mathbf{Ban}_R))$.

Idea of proof: Projective objects of $\mathbf{Born}_R$ and of $\mathbf{Ind}(\mathbf{Ban}_R)$ are the same and the tensor product of projectives is projective.

Schneiders proved it for $R = (\mathbb{C}, |\cdot|_\infty)$.

Notice that $\mathbf{Born}_R \subset \mathbf{Ind}(\mathbf{Ban}_R)$ is not strong monoidal.

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Some derived functors

I give some examples of computations of derived functors.

Let $M \in \mathbf{Born}_R$, then there is a projective resolution $P^\bullet \to M$ and for any $N \in \mathbf{Born}_R$

$$N \widehat{\otimes}^{\mathbb{L}}_R M \cong N \widehat{\otimes}_R P^\bullet.$$

This computation is possible, but often it is easier to work with flat resolutions.

Flat object

A module M is said to be *flat* if the functor $(\cdot)\widehat{\otimes}_R M$ is strongly exact.

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Flat bornological modules

Proposition

Projective modules are flat.

Let k be a non-Archimedean valued field. Then, all objects of $\mathbf{Ban}_k^{\mathrm{na}}$ are flat.

It follows that:

- all objects of $\mathbf{Born}_k^{\mathrm{na}}$ are flat.
- for any ultrametric Banach k-algebra A, the Tate algebras $A\langle T_1, \ldots, T_n\rangle$ are flat (they are actually projective).

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Long exact sequence Tor

Now, let

$$0 \to N \to M \to L \to 0$$

be a strict exact sequence in $\mathbf{Born}_R$.

In particular this is a distinguished triangle in $D(\mathbf{Born}_R)$. Hence

$$N \widehat{\otimes}_R^{\mathbb{L}} M' \to M \widehat{\otimes}_R^{\mathbb{L}} M' \to L \widehat{\otimes}_R^{\mathbb{L}} M' \to +$$

is a distinguished triangle too for any M'.

This implies that we have a long exact sequence for the left-heart cohomology

$$\cdots \longrightarrow \mathrm{LH}^i(N \widehat{\otimes}_R^{\mathbb{L}} M') \longrightarrow \mathrm{LH}^i(M \widehat{\otimes}_R^{\mathbb{L}} M') \longrightarrow \mathrm{LH}^i(L \widehat{\otimes}_R^{\mathbb{L}} M') \longrightarrow$$

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If the differentials of $N \widehat{\otimes}_R^{\mathbb{L}} M'$, $M \widehat{\otimes}_R^{\mathbb{L}} M'$, $L \widehat{\otimes}_R^{\mathbb{L}} M'$ are strict morphisms then these cohomology groups are bornological spaces.

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Other derived functors

Proposition

- For any set I the functor $\mathbf{Born}_k^I \to \mathbf{Born}_k$ given by $(M_i) \mapsto \prod_{i\in I} M_i$ is strongly exact.
- For any directed set I the functor $\mathbf{Born}_k^I \to \mathbf{Born}_k$ given by $(M_i) \mapsto \varinjlim_{i\in I} M_i$ is strongly exact.

It follows that $\prod_{i\in I}$ and $\varinjlim_{i\in I}$ derive trivially.

The derived functor $\mathbb{R}\varprojlim_{i\in I}$ can be computed using the Roos complex.

Mittag-Leffler lemma for Fréchet spaces

If $(E_n)_{n\in\mathbb{N}}$ is a projective system of (nuclear) Fréchet spaces such that $E_{n+1} \to E_n$ is dense then in $D^+(Born_k)$

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If $(E_n)_{n\in\mathbb{N}}$ is a projective system of (nuclear) Fréchet spaces such that $E_{n+1} \to E_n$ is dense then in $D^+(Born_k)$

$$\mathbb{R}\varprojlim_{n\in\mathbb{N}} E_n \cong \varprojlim_{n\in\mathbb{N}} E_n.$$

Compact objects

Compact object

$M \in \mathbf{Born}_R$ is *compact* if $\mathrm{Hom}\,(M, \cdot)$ commutes with filtered colimits.

Proposition

Let R be a Banach ring, then $M \in \mathbf{Born}_R$ is compact if and only if it is a Banach module.

A cocomplete quasi-abelian category with strongly exact filtered colimits and with a set of compact generators is called *elementary*.

For any bornological algebra $\mathbf{Born}_R$ is elementary.

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Simplicial bornological algebras

We will consider the category $\mathbf{SComm}(\mathbf{Born}_R)$ of simplicial bornological algebras.

There is a symmetric monoidal model structure on $\mathbf{SComm}(\mathbf{Born}_R)$ and we will consider objects of $\mathbf{Comm}(\mathbf{Born}_R)$ as discrete objects of $\mathbf{SComm}(\mathbf{Born}_R)$.

Dold-Kan correspondence

For any bornological algebra one has the Dold-Kan equivalence

$$N : \mathbf{SBorn}_R \leftrightarrows \mathbf{Ch}^{\geq 0}(\mathbf{Born}_R) : \Gamma$$

Derived Dold-Kan correspondence

On $\mathbf{SBorn}_R$ there is a model structure such that weak equivalences correspond to chain morphism whose cone is strictly exact after applying the functor N.

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Thanks

Thanks for your attention.