Tempered functions in derived analytic geometry

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- Ban_R will be the category of nonarchimedean Banach modules over R with bounded morphisms.
- CBorn_R will be the category of *complete bornological modules* over R.

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Example

The polynomial algebra $R[t]$ is an element of CBorn_R.

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Let $n \in \mathbf{N}$, we denote by $R[\![t]\!]_n$ the Banach module of power series with *log-growth* bounded by n:

$$\left\{ \sum_{i \in \mathbf{N}} a_i t_i \in R[\![t]\!] : a_i \in R,\ \sup_{i \in \mathbf{N}} |a_i|_R (i+1)^{-n} < \infty \right\}.$$

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These are the *tempered functions* over R. They form an algebra in CBorn_R.

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Let $\mathrm{Comm}(\mathcal{C})$ be the ∞-category of commutative algebras in $\mathcal{C}$. A morphism $A \to A' \in \mathrm{Comm}(\mathcal{C})$ is called *homotopy epimorphism* if

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Let $\mathfrak{I}(\mathcal{C})$ be the set of homotopy epimorphisms of the form $1 \to A$, for $A \in \mathcal{C}$.

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This picture is "dual" to the classical point of view of homotopical algebraic geometry.

The spectrum of a closed symmetric monoidal stable ∞-category

Using the duality between distributive lattice and coherent spaces (Johnstone) we constructed a spectral space $\mathfrak{S}(\mathcal{C})$ such that $\mathfrak{I}(\mathcal{C})$ corresponds to a basis of compact open subsets of $\mathfrak{S}(\mathcal{C})$.

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This space is naturally endowed with a structure sheaf $\mathcal{O}_{\mathfrak{S}(\mathcal{C})}$ with values in $\mathrm{Comm}(\mathcal{C})$, defined by

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where $A \in \mathfrak{I}(\mathcal{C})$ and U_A is the open set in $\mathfrak{S}(\mathcal{C})$ corresponding to A.

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The space $\mathfrak{S}(\mathcal{C})$ recovers the (formal) homotopical Zariski Grothendieck topology on $\mathrm{Comm}(\mathcal{C})$.

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If $R = K$, $K[t] \to K\langle t \rangle$ and affinoid localizations are homotopy epimorphisms in CBorn_K (Ben Bassat-Kremnizer and Ben Bassat-Mukherjee) $\implies$ this picture "refines" rigid analytic geometry.

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Proposition

The inclusion

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Proposition

The inclusion

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is an homotopy epimorphism in $\mathrm{Comm}(\mathcal{D}(\mathrm{CBorn}_R))$.

In particular $R[\![t]\!]_{\mathrm{temp}}$ defines an open *tempered unit ball*: $\mathbf{B}^1_{\mathrm{temp}}(1)$ in $\mathbf{A}^{1,\mathrm{der}}_R$.

The derived affine line over R

We have for example that

$$\mathbf{B}^1_{\mathrm{open}}(1) \subset \mathbf{B}^1_{\mathrm{temp}}(1) \subset \mathbf{B}^1_{\mathrm{bound}}(1) \subset \mathbb{B}^1(1),$$

where $\mathbf{B}^1_{\mathrm{open}}(1)$, $\mathbf{B}^1_{\mathrm{bound}}(1)$ and $\mathbb{B}^1(1)$ are the open unit balls in $\mathbf{A}^{1,\mathrm{der}}_R$ associated respectively to $R\{\{t\}\}$ (the series convergent in $|\cdot|_R < 1$), $R[\![t]\!]_0$ and $R\langle t\rangle$ (they all define homotopy epimorphisms from the unit).

Application: Tempered transfer theorem

Let

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} = G\mathbf{y} \tag{1}$$

be a differential system over $K\langle t\rangle$ ($G \in \mathrm{M}_d(K\langle t\rangle)$).

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be a differential system over $K\langle t\rangle$ ($G \in \mathrm{M}_d(K\langle t\rangle)$).

By formal Cauchy's theorem the differential system (1) admits a full set of formal solutions $\mathcal{Y} \in \mathrm{GL}_d(K[\![t]\!])$, $d \in \mathbf{N}$. If $K = \mathbb{C}$ solutions converge everywhere. But not in the p-adic world!

Application: Tempered transfer theorem

Let w be a formal variable. Functions over K can be developed at w (the "generic point") via the morphism of differential rings

$$\tau : K\langle t\rangle \to K\langle t\rangle[\![w]\!]_0,$$

$$\tau : f(t) \mapsto \sum_i \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i \frac{f}{i!}\right) w^i.$$

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We obtain a new differential system

$$\frac{\mathrm{d}}{\mathrm{d}w}\mathbf{y} = \tau(G)\mathbf{y}. \qquad (2)$$

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The radius of convergence of the formal solutions of the differential system (1) is bounded below by the radius of convergence of the development at the generic point.

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In Berkovich spaces the generic point corresponds to the point given by the Gauss norm $\Longrightarrow$ this theorem can be interpreted as a continuity result (Baldassarri, Poineau-Pulita).

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Theorem (Christol)

If the development of $\mathcal{Y}$ is in $\mathrm{GL}_d(K\langle t\rangle[\![w]\!]_{\mathrm{temp}})$ then $\mathcal{Y} \in \mathrm{GL}_d(K[\![t]\!]_{\mathrm{temp}})$.

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The theorem above can be interpreted as a continuity theorem in our framework.

Application: Tempered transfer theorem

We have the commutative diagram:

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\begin{array}{ccc}
 & \mathbf{A}_K^{2,\mathrm{der}} & \\
 \swarrow^{\tau} & & \nwarrow^{t=0} \\
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In particular if the solutions of (2) live in the open $\mathbf{B}^2_{\mathrm{temp}}(1) \subset \mathbf{A}_K^{2,\mathrm{der}}$, by continuity of the map $t = 0$, $\mathcal{Y}$ lives in the inverse image open $\mathbf{B}^1_{\mathrm{temp}}$.

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Example

Consider $X_k = \{pt\} \hookrightarrow \mathbf{A}^{1,\mathrm{an}}_{\mathcal{V}}$; here the tube is $\mathbf{B}^1_{\mathrm{open}}(1)$. The cohomology is computed by

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But we can take $\mathbf{B}^1_{\mathrm{temp}}(1)$ instead of $\mathbf{B}^1_{\mathrm{open}}(1)$ and we obtain the same.

Thank you for your attention!